Published by Institute of Physics Publishing for SISSA/ISAS
Received: *September 23, 2003*
Accepted: *October 10, 2003*

A note on domain walls and the parameter space of $\mathcal{N}=1$ gauge theories

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Abstract: We study the spectrum of BPS domain walls within the parameter space of $\mathcal{N}=1$ U(N) gauge theories with adjoint matter and a cubic superpotential. Using a low energy description obtained by compactifying the theory on $\mathbb{R}^3 \times S^1$, we examine the wall spectrum by combining direct calculations at special points in the parameter space with insight drawn from the leading order potential between minimal walls, i.e those interpolating between adjacent vacua. We show that the multiplicity of composite BPS walls — as characterised by the CFIV index — exhibits discontinuities on marginal stability curves within the parameter space of the maximally confining branch. The structure of these marginal stability curves for large-N appears tied to certain singularities within the matrix model description of the confining vacua.

Keywords: Nonperturbative Effects, Solitons Monopoles and Instantons, Brane Dynamics in Gauge Theories, Supersymmetry and Duality.

Contents

1. Introduction

Four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories are believed, and in some cases have been shown, to exhibit many of the subtle and physically relevant phases seen within the standard model. In particular, in the last decade technical advances have allowed us to explore properties such as abelian confinement and chiral symmetry breaking within specific $\mathcal{N} = 1$ theories [1, 2]. However, this progress has generally been limited to particular examples, and one may hope that a more global perspective on the space of $\mathcal{N} = 1$ theories is attainable. In this regard, interesting recent work by Ferrari [3] and Cachazo, Seiberg and Witten [4, 5] has provided insight into the structure of the quantum parameter space of $\mathcal{N} = 1$ theories. This work was stimulated by the realisation of Dijkgraaf and Vafa [6] that a matrix model structure apparently underlies the chiral sector of $\mathcal{N} = 1$ gauge theories, and specifically those with adjoint matter.

For a theory with gauge group $\mathrm{U}(N)$, and a superpotential $\mathrm{tr}\,\mathcal{W}(\Phi)$ for the adjoint chiral superfield Φ, the parameter space in question corresponds to the set of dimensionless variables, modulo symmetries, that one can construct based on the parameters in $\mathcal{W}(\Phi)$

and the dynamically generated SU(N) scale $\Lambda_{N=2}$. We will focus on the simplest nontrivial example, with a cubic superpotential,

$$\mathcal{W}_{\rm cl}(\Phi) = \frac{1}{2} m\,\Phi^2 + \frac{1}{3} g\,\Phi^3\,. \tag{1.1}$$

Classically this theory has two vacua within which the gauge group is unbroken, which is the phase on which we will focus. In the limit that $g = 0$, the infrared theory is simply $\mathcal{N} = 1$ super Yang-Mills (SYM) and the dependence of $\mathcal{W}|_v$ on $m/\Lambda_{N=2}$ is fixed by a decoupling relation so that only a single fractional instanton can contribute in the confining vacua of the unbroken SU(N). More generally, for finite g one finds a nontrivial sum of fractional instanton contributions, leading to an effective superpotential of the form [3],

$$\mathcal{W}_k = \frac{2N}{3\lambda}\Lambda^3_{N=1}\left(1 \pm \left[1 - \lambda e^{2\pi i k/N}\right]^{3/2}\right), \quad k = 0, \ldots, N-1\,, \tag{1.2}$$

in terms of the dimensionless parameter,

$$\lambda = \frac{8g^2\Lambda^2_{N=2}}{m^2}\,, \tag{1.3}$$

which is a convenient coordinate [3] for the parameter space associated with the confining vacua (1.2).

These vacua lie on the maximally confining branch where the only remaining massless field is a decoupled U(1) multiplet. However, this branch connects to others associated with classical vacua for which the gauge group is partially broken. It has been conjectured that all these transitions are of two basic types [3]: (1) where there are additional massless monopoles, such as the Seiberg-Witten singularities connecting Coulomb and Higgs branches in $\mathcal{N} = 2$ SYM; and (2) where the gluino condensate vanishes corresponding to branch points in (1.2). These two cases may alternatively be characterised as the result of summation over instanton and respectively fractional-instanton contributions. In the case of (1.2), the branch points at [7, 3]

$$\lambda_k = e^{-2\pi i k/N} \tag{1.4}$$

also imply, for N even, the presence of massless monopoles, and thus play a dual role. These points were first noted in the latter context, namely as singular points connecting Coulomb and confining branches, by Argyres and Douglas for SU(3) [7]. Furthermore, in addition to the presence of these singularities connecting branches in different phases, it was shown by Cachazo, Seiberg and Witten [4] that particular phases may also exhibit multiple classical limits, connected by smooth transitions through strong coupling regions in parameter space. For the cubic model (1.1), these smooth transitions always correspond to massless branches with $N \geq 4$.

The presence of a connected set of vacuum branches, in general describing a multi-sheeted covering of the quantum parameter space, suggests that it may be profitable to study the action of symmetries on this space. To this end, the aim of this paper is to go beyond the vacuum structure and explore the spectrum of 1/2-BPS states within the

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parameter space. For the $\mathcal{N}=1$ theories studied here, these BPS states are domain walls connecting discrete massive vacua, for which the data which can be extracted from the chiral sector of the theory is limited to the tension, determined immediately via the vacuum condensates, and also the multiplicity. The latter degeneracy is formally determined by the CFIV index [8] in the dimensionally reduced 1+1D theory obtained by compactifying the worldvolume dimensions of the wall on a 2-torus. For our purposes, the CFIV index will be understood to provide a definition of the wall multiplicity.

To determine the multiplicity of BPS states we require a Wilsonian low energy description, and since the vacua of interest are confining this requires some deformation of the original theory. We find that compactifying the theory on a circle of radius R, and using the known relation to integrable systems [9, 10], is useful for this purpose. This approach also provides a straightforward means of reproducing the vacuum structure, as first utilised by Dorey for the $\mathcal{N}=1^*$ theory [11], and used recently by Boels et al. [12] to determine the vacuum condensates for $\mathcal{N}=1$ models with adjoint fields, such as the example studied here.

We explore the multiplicity of certain BPS walls connecting confining vacua as a function of the relevant parameter λ, and find that the spectrum exhibits discontinuities on curves of marginal stability (CMS), with certain bound states excluded from compact domains in the λ-plane. The data describing the multiplicities of BPS states thus form nontrivial sections over the parameter space, in a rather close analogy to the way BPS particle multiplets form sections, transforming under subgroups of $\mathrm{SL}(2,\mathbb{Z})$, over the moduli space of $\mathcal{N}=2$ gauge theories. While we do not yet see evidence for nontrivial quantum symmetries of the latter form, we hope that further analysis in this direction will lead to novel constraints.

The plan of the paper is as follows. In the next section, we describe the compactification of the theory on $\mathbb{R}^3\times S^1$, which allows a low energy effective superpotential to be obtained. This arises from instantons and thus has a natural semiclassical interpretation. Using this effective theory, we reproduce the condensates (1.2) in the maximally confining vacua. In sections 3, we turn to BPS walls and study the structure of the central charges as a function of λ, and exhibit a class of marginal stability curves for composite states. The structure of these curve simplifies for large-N and has an interesting connection with certain singularities within the relevant Dijkgraaf-Vafa matrix model, which we also describe. In section 4, we turn to the dynamical question of whether these marginal stability curves do indeed signify discontinuities in the spectrum, and verify that this is the case by first deducing the spectrum at a special point — $\lambda=0$ — and then constructing the leading order inter-wall potential as a function of λ. This leads to the conclusion that specific composite states are removed from the spectrum within a compact domain of the λ-plane. Section 5 contains some concluding remarks, along with further comments on the embedding of this theory within $\mathcal{N}=1^*$ SYM.

2. Compactification and the vacuum structure

The class of theories we will focus on here contains an $\mathcal{N}=1$ vector multiplet with gauge group $\mathrm{U}(N)$ and an adjoint chiral multiplet Φ, with scalar component ϕ. The cubic superpotential

$$\operatorname{tr}\mathcal{W}_{\rm cl}(\Phi)=\frac{1}{2}m\operatorname{tr}\Phi^2+\frac{1}{3}g\operatorname{tr}\Phi^3 \tag{2.1}$$

leads to a set of classical vacua, distinguished by the distribution of the N eigenvalues φ^a of ϕ between the two minima,

$$\varphi_{(1)}=0\,,\qquad \text{and}\qquad \varphi_{(2)}=-\frac{m}{g}\,. \tag{2.2}$$

With N_1 eigenvalues equal to $\varphi_{(1)}$ and N_2 equal to $\varphi_{(2)}$, satisfying $N_1+N_2=N$, the gauge group is broken to $\mathrm{U}(N_1)\times\mathrm{U}(N_2)$. We will be concerned primarily with those vacua where the gauge group is classically unbroken, i.e. with (N_1,N_2) equal to $(N,0)$ and $(0,N)$. Note that the overall $\mathrm{U}(1)$ factor of the gauge group is central and thus decouples as all fields are correspondingly uncharged.

Our aim in this section will be to construct a low energy effective superpotential suitable for use in extracting the BPS spectrum. This will necessarily involve a deformation of the theory. However, to first deduce the quantum vacuum structure on the confining branch, we can proceed more directly. Treating this system as a perturbation of $\mathcal{N}=2$ SYM, the vacuum structure follows from the Seiberg-Witten solution. i.e. given $\mathcal{W}_{\rm cl}$, we can write

$$\mathcal{W}_{\rm eff}=\langle\operatorname{tr}\mathcal{W}_{\rm cl}(p\,\mathbb{I}+\hat{\Phi})\rangle=N\mathcal{W}_{\rm cl}(p)+\frac{1}{2}\mathcal{W}_{\rm cl}^{(2)}(p)\langle\operatorname{tr}\hat{\Phi}^2\rangle\,, \tag{2.3}$$

where $\hat{\Phi}$ transforms in the adjoint of $\mathrm{SU}(N)$, and so we have made use of the relations $\langle\operatorname{tr}\hat{\Phi}^{2n+1}\rangle=0$, while $p=\langle\operatorname{tr}(\Phi)\rangle/N$. For confining vacua, the maximal degeneration of the $\mathrm{SU}(N)$ Seiberg-Witten curve [13, 14] leads to the non-vanishing condensate [15] $\langle\operatorname{tr}\hat{\Phi}^2\rangle=2N\omega_k\Lambda^2_{\mathcal{N}=2}$ with $\omega_k=\exp(2\pi ik/N)$ an N^{th}-root of unity. On substitution into (2.3), one finds [3, 16]

$$\mathcal{W}_k(p)=N\left[\mathcal{W}_{\rm cl}(p)+m(p)\omega_k\Lambda^2_{\mathcal{N}=2}\right]\,,\qquad k=0,\ldots,N-1\,, \tag{2.4}$$

where $m(p)=\mathcal{W}''_{\rm cl}(p)$ is the effective mass term for all the scalar modes, including the trace component p, at weak gauge coupling. For the cubic case this result, first obtained in [3, 16], has the simple interpretation of contributing the gluino condensate from the confining $\mathrm{SU}(N)$ factor. More generally, one expects a nontrivial (but finite) sum of such fractional instanton contributions.[1] Integrating out the trace component p, we recover the confining vacua (1.2) noted earlier.

When the trace component p is light relative to the other modes – and indeed it becomes massless when $\lambda=\lambda_k$ — the effective superpotential (2.4) may provide a reliable low energy description. However, since (2.4) depends explicitly on the vacuum, we cannot use it to study wall configurations which interpolate between different vacua. These configurations must necessarily excite the gauge modes responsible for gluino condensation, and we will need to include them in any tractable low energy description. Moreover, since the vacua of interest are confining, some deformation of the theory is required. To this end, if we

[1] In certain cases, there may also be additional instanton contributions, whose form determines the ultraviolet completion of the theory. See [17] for a nice discussion of this issue.

wish to avoid adding additional fields, a natural approach is to compactify the theory on a circle of radius R. A Wilson line for the gauge field then provides a massless scalar, which may be rotated to the Cartan subalgebra $\rho_a = \int_{S^1} A_a$, $a = 1 \dots N$, and this will generically Higgs the gauge group to its maximal torus. The resulting photons can be dualised to periodic scalars σ^a [18], and supersymmetry then ensures that the complex combination,

$$\tilde{q}_a = \rho_a + \tau \sigma_a \,, \qquad a = 1, \dots, N \,, \tag{2.5}$$

where τ is the complex gauge coupling, is the lowest component of a (classically massless) chiral superfield which characterises the gauge sector of the dimensionally reduced low energy theory. The additional adjoint scalars reduce straightforwardly, and the effective 3D theory is a Wess-Zumino model with two adjoint chiral fields and a classical superpotential given by (2.1). We now turn to possible nonperturbative corrections.

2.1 Semiclassical interpretation of $\mathcal{W}_{\rm eff}$

When the compactification radius R is sufficiently small, $\Lambda R \ll 1$, the system is weakly coupled and thus any nonperturbative corrections to $\mathcal{W}$ should have a semiclassical interpretation in terms of instantons. Therefore, we can parametrise the superpotential in the form

$$\mathcal{W}_{\rm eff} = \mathcal{W}_{\rm cl}(p_a) + \mathcal{W}_{\rm inst}(q_a, p_a) \,, \tag{2.6}$$

where $\mathcal{W}_{\rm class}(p_a)$ is the classical superpotential (2.1) and we have now chosen to denote the eigenvalues of the adjoint field Φ as $\{p_a\}$, $a = 1, \dots, N$.

The form of the instanton generated superpotential in these theories was first explored by Polyakov [18], and for theories with $\mathcal{N} = 2$ SUSY by Affleck, Harvey and Witten [19]. For pure $\mathcal{N} = 1$ SYM, there are N nontrivial 1-instanton configurations, corresponding to $N - 1$ BPS monopoles aligned along simple roots and, for finite R, an additional 'Kaluza-Klein'-monopole wrapped around the compact direction [20], which contribute to the superpotential [21, 22, 23, 24, 25, 11, 26, 27, 28]. In the presence of additional adjoint chiral fields, these configurations would normally have too many fermionic zero modes to contribute to $\mathcal{W}$, but if these fields are classically massive, the additional zero modes can be absorbed by the mass terms. Accounting for the monopole-antimonopole background [18] which resums the instanton vertices to a field-dependent superpotential, we can write down its generic form as follows [26, 27],[2]

$$\mathcal{W}_{\rm inst} = \frac{1}{2}\Lambda^2_{\mathcal{N}=2} \sum_{a=1}^{N} [m(p_a) + m(p_{a+1})] e^{q_a - q_{a+1}} \,, \qquad m(p_a) = m + 2g p_a \,, \tag{2.7}$$

in which $m(p_a)$ is the classical mass term for φ_a given by expanding $\mathcal{W}_{\rm cl}$ about a given vacuum $\langle p_a \rangle$. Note that q_a and $\tilde{q}_a$ in (2.5) are related by a constant shift depending on the coupling τ. The structure of the mass insertions follows from the choice of basis for the scalars which corresponds to taking projections along the fundamental weights. The monopoles in contrast are aligned along the simple roots, and thus are embedded along two fundamental weight directions.

[2] For simplicity, we will write the superpotential using a 3+1D normalisation in what follows, so that $\mathcal{W}_{\rm 3D} = 2\pi R\, \mathcal{W}_{\rm eff}$.

In the expression (2.7), we have slightly extended the result of Davies et al. [26, 27], having made the replacement $\langle p_a \rangle \to p_a = \langle p_a \rangle + \delta p_a$, so that a genuine interaction term is introduced. We have not derived the appropriate instanton vertex explicitly, as it can be justified in a different manner to be discussed shortly. In particular, for later use, it is convenient to recall another approach to this problem which starts from the $\mathcal{N}=2$ limit and uses the relation to integrable systems [11].

2.2 The effective Toda superpotential

At a formal level, the calculation of $\mathcal{W}_{\text{eff}}$ is dramatically simplified by recalling the relation between the Seiberg-Witten solution for $\mathcal{N}=2$ SYM and certain (complexified) integrable systems [9, 10]. More precisely, written in terms of the natural gauge invariant coordinates on the $\mathcal{N}=2$ moduli space,

$$u_n \equiv \frac{1}{n} \operatorname{tr} \phi^n \,, \tag{2.8}$$

the superpotential takes the form

$$\operatorname{tr} \mathcal{W} = m u_2 + g u_3 \,, \tag{2.9}$$

where we have now dropped the subscript as one can argue that this form is exact within the compactified theory on $\mathbb{R}^3 \times S^1$. To do this, one treats this superpotential as a perturbation of the (reduced) $\mathcal{N}=4$ theory, and interprets the coordinates $u_n = u_n(x)$ as constrained variables depending on the underlying Seiberg-Witten curve. To make use of this expression one must find the corresponding unconstrained variables parametrising the jacobian of the curve.

This problem is elegantly solved via the correspondence with integrable systems, and in this case the affine Toda lattice [9]. This relation is perhaps most transparent on noting that the Seiberg-Witten curve for $\mathrm{U}(N)$ can be written in the form,

$$P_N(x, u_n) = z + \frac{\Lambda_{N=2}^{2N}}{z} \,, \qquad P_N(x, u_n) = x^N + \sum_{n=1}^{N} s_n x^{N-n} \,, \tag{2.10}$$

where s_n and u_n are related by the recursion relation, $r s_r + \sum_{i=0}^{r} s_{r-i} u_i = 0$, with $u_0 = 0$ and $s_0 = 1$. This curve is identifiable as the spectral curve for the affine Toda lattice, namely

$$\det\left(x\mathbb{I} - L(z)\right) = 0 \,, \tag{2.11}$$

where the Lax matrix is given by

$$L(z) = \begin{pmatrix} p_1 & \Lambda_{N=2}\, e^{q_1 - q_2} & 0 & \cdots & z \\ 1 & p_2 & \Lambda_{N=2}\, e^{q_2 - q_3} & \cdots & 0 \\ 0 & 1 & p_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ z^{-1}\Lambda_{N=2}\, e^{q_N - q_1} & 0 & \cdots & 1 & p_N \end{pmatrix} \tag{2.12}$$

in terms of the canonical variables $\{p_a, q_a\}$. Thus the unconstrained variables on the jacobian of the curve are precisely the (complex) coordinates and momenta $\{q_a, p_a\}$ for the

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affine Toda lattice. The conserved quantities, to be identified with u_n, are given by

$$u_n = \frac{1}{n}\,\mathrm{tr}\, L^n\,, \tag{2.13}$$

which allows us to identify the momenta p_a with the eigenvalues of the adjoint scalar $p_a = \langle\varphi_a\rangle$, explaining the notation introduced above.

With this parametrisation, the superpotential (2.9) takes the form

$$\mathcal{W}_{\rm eff} = \frac{1}{2}m\left[\sum_{a=1}^{N} p_a^2 + 2\Lambda_{\mathcal{N}=2}^2 e^{q_a - q_{a+1}}\right] + \frac{1}{3}g\left[\sum_{a=1}^{N} p_a^3 + 3\Lambda_{\mathcal{N}=2}^2 (p_a + p_{a+1}) e^{q_a - q_{a+1}}\right] \tag{2.14}$$

which, on rewriting it in the form (2.6), agrees with the expected 1-instanton corrected potential.[3]

This latter approach for calculating $\mathcal{W}_{\rm eff}$, making use of the connection to integrable systems, was first used by Dorey for the $\mathcal{N} = 1^*$ theory, and has since been applied to other theories including pure $\mathcal{N} = 1$ SYM [28, 27] and more general deformations $\mathcal{W}_{\rm cl}$ [29, 30]. A wider class of U(N) examples involving polynomial superpotentials was also recently studied in this way by Boels et al. [12], and subsequently extended to other gauge groups [31].

2.3 Quantum vacuum structure

Extremising the superpotential (2.14), we find the following equations,

$$p_a(\eta p_a + 1) = -\eta(x_a + x_{a+1}) \tag{2.15}$$

$$(1 + \eta(p_a + p_{a+1}))\, x_a = (1 + \eta(p_{a-1} + p_a))\, x_{a-1}\,, \tag{2.16}$$

where $\eta = g/m$ and $x_a = \exp(q_a - q_{a+1})$, and we have imposed the decoupling condition $q_{N+1} = q_1$ for the overall U(1) factor. Since we are interested in vacua with a classically unbroken gauge group, we solve (2.16) by setting

$$q_a = q\,, \quad p_a = p\,, \quad \forall a = 1, \ldots, N\,. \tag{2.17}$$

The resulting $2N$ vacua are given by

$$u_1 = \sum_a p_a = N p_k^{\pm} = \frac{N}{2\eta}\left(1 \mp \sqrt{1 - \lambda x_k}\right), \tag{2.18}$$

with

$$x_k = e^{2\pi i k/N}\,, \qquad k = 0, \ldots, N-1\,, \tag{2.19}$$

where u_1, and thus p, is single valued on a double-sheeted cover of the λ-plane, and the two sheets are distinguished by the corresponding classical vacuum expectation value for ϕ. One may note here that the intuitive correspondence between massive vacua and (complex)

[3] Note that if we were to consider a classical superpotential of degree four or higher, e.g. $\mathrm{tr}\,\Phi^4$, then the semiclassical interpretation would require the contribution of certain multi-instanton corrections. It would be interesting to see precisely how the additional zero modes are lifted in such cases.

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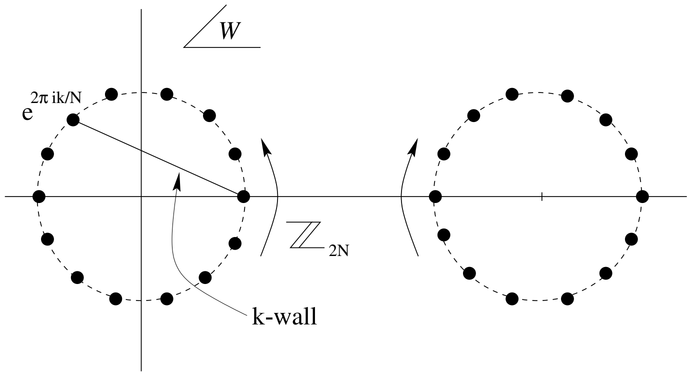

Figure 1: Confining vacuum structure in the $\mathcal{W}$-plane for $N = 14$ and $\lambda \ll 1$, exhibiting the action of the cyclic '$\mathbb{Z}_{2N}$ symmetry', and illustrating the interpolating profile of a k-wall.

mechanical equilibria of the integrable system, that is so apparent for SU(N) [11], appears to have been lost here due to the nontrivial vacuum expectation value for p, so that the 'equilibria' now have nonzero coordinate momentum. However, one finds that these vacua still correspond to stationary points where the angular momentum within the jacobian vanishes [32, 33]. In other words, there is a canonical transformation within the integrable system to an action-angle basis for which the massive vacua still map to mechanical equilibria.

The $2N$ extrema of the superpotential are then given by,

$$\mathcal{W}_k^{\pm} = \frac{2N}{3\lambda}\Lambda_{\mathcal{N}=1}^3\left(1 \mp [1-\lambda x_k]^{3/2}\right), \qquad k = 0, \ldots, N-1, \tag{2.20}$$

and these results are fully consistent with those recently obtained using Seiberg-Witten and matrix model techniques [3, 16] as discussed above.

For $\lambda \ll 1$ the vacua fall on two approximately circular curves in the $\mathcal{W}$-plane, as exhibited in figure 1. For the vacua corresponding to the classical vacuum at $\phi = 0$, the asymptotics near $\lambda = 0$ have the form

$$\mathcal{W}_k^{+}(\lambda \to 0) \sim N\Lambda_{\mathcal{N}=1}^3 e^{2\pi i k/N}\left(1 + \mathcal{O}\left(g^2\left(\frac{\Lambda_{\mathcal{N}=1}}{m}\right)^3\right)\right), \tag{2.21}$$

where the leading term corresponds to the pure $\mathcal{N} = 1$ SYM limit, and the subleading terms, proportional to powers of $e^{2\pi i\tau(m)/N}$, correspond to higher order fractional instantons.

The plot also exhibits a discrete cyclic symmetry which rotates the vacua via its action on λ,

$$\lambda \to e^{2\pi i/N}\lambda, \tag{2.22}$$

which we interpret as induced by SL$(2, \mathbb{Z})$ translations on the bare gauge coupling $\tau \to \tau+1$. The corresponding action on the vacua within each branch is then

$$\mathcal{W}_k^{\pm} \to \mathcal{W}_{k+1}^{\pm}, \tag{2.23}$$

while the rotation of λ around one of the branch points induces the action [3]

$$\mathcal{W}_k^{\pm} \to \mathcal{W}_k^{\mp}\,. \tag{2.24}$$

With a suitable choice of branch, the action (2.22) rotates $\mathcal{W}_k$ through all $2N$ vacua on both branches, and we can interpret this '$\mathbb{Z}_{2N}$ cyclic symmetry' as a nontrivial extension of the $\mathbb{Z}_N$ nonanomalous discrete subgroup of the classical $\mathrm{U}(1)_R$ symmetry, which rotates the vacua of pure SYM. The extension arises directly from the double-sheeted structure of the confining branch over the quantum parameter space, coordinatised by λ.

Note that this structure does not, however, extend globally over the parameter space. In particular, the asymptotics near $\lambda = \infty$ are

$$\mathcal{W}_k^{\pm}(\lambda \to \infty) \sim \pm\frac{4\sqrt{2}i}{3} N g \Lambda_{N=2}^3 e^{3\pi i k/N}\left(1 + \mathcal{O}\left(\frac{1}{g^2}\left(\frac{m}{\Lambda_{N=2}}\right)^2\right)\right). \tag{2.25}$$

and λ is no longer a good coordinate on the parameter space – the leading dependence is instead on $\sqrt{\lambda}$.

We have focused on the maximally confining vacua, but for completeness we note that for generic N there are in addition a large number of branches with massless vacua which map to classical vacuum solutions with a broken gauge group in the limit that $\Lambda_{N=2} \to 0$. The cubic superpotential only exhibits classical vacua where the the eigenvalues $\{p_a\}$ are split into two groups, and thus there is a single (nontrivial) massless $\mathrm{U}(1)$ factor. The structure of these vacua was elaborated for $N \leq 6$ in [4], and shown to include branches where smooth transitions in parameter space between distinct classical limits were possible. These vacua decouple in the limit that $\lambda \to 0$, where the theory reduces in the infrared to pure SYM. However, in the opposite limit, $\lambda \to \infty$, certain massless vacua survive and in the $\mathrm{U}(3)$ case, for example, apparently asymptote to the Argyres-Douglas conformal points [7].

3. BPS wall kinematics

With a clear picture of the confining vacuum structure in the $\mathcal{W}$-plane, we can now turn to the question of the spectrum of BPS states, namely domain walls interpolating between distinct vacua. Given that the vacua described above are distinct, such configurations are necessarily present on topological grounds. However, the number of corresponding BPS states is a dynamical question that we will come to shortly. We first outline the kinematic structure imposed by the $\mathcal{N} = 1$ superalgebra.

3.1 Central charges

The vacua labelled by $\mathcal{W}_k$ allow us to isolate two sets of walls, characterised by whether or not both vacua at spatial plus and minus infinity are on the same branch. If they are, then such walls are present only at the quantum level. However, if the asymptotic vacua are on different branches then the corresponding walls are visible classically. More precisely, as noted in [3], this characterisation is useful for $\lambda \ll 1$ while, as we will see, by varying λ one

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observes various discontinuities in the spectrum of BPS walls, with the underlying reason being the presence of curves of marginal stability in the parameter space coordinatised by λ.

We begin by discussing the central charge structure, and to this end it is useful to introduce the following (condensed) notation for the vacua,

$$\mathcal{W}_k = \frac{2N}{3\lambda}\Lambda^3_{N=1}\left(1 - [1-\lambda x_k]^{3/2}\right), \quad k = 0, \dots, 2N-1\,, \tag{3.1}$$

which incorporates both branches, with the implicit understanding that the vacua for $k = N, \dots, 2N-1$ lie on the second branch. This notation is consistent with the discrete $\mathbb{Z}_{2N}$ symmetry noted above. It is then convenient to introduce the (SU(N) root-valued) topological charges,

$$n^k_{(ij)} = \delta^k_j - \delta^k_i\,, \tag{3.2}$$

in terms of which the central charges for walls interpolating between vacua labelled by i and j phase-units respectively are given by [34]

$$\mathcal{Z}_{ij} = 2n^k_{(ij)}\left[\mathcal{W}_k - \frac{1}{16\pi^2}\left(T_G - \sum_f T(R_f)\right)\langle \mathrm{tr}\, W^\alpha W_\alpha\rangle_k\right]. \tag{3.3}$$

The second term here is an anomaly, which vanishes in the present case due to the $\mathcal{N}=2$ matter content. Consequently, we have

$$\mathcal{Z}_{ij} = 2n^k_{(ij)}\mathcal{W}_k = 2\left(\mathcal{W}_j - \mathcal{W}_i\right), \tag{3.4}$$

or more explicitly,

$$\mathcal{Z}_{jk} = \frac{4N}{3\lambda}\Lambda^3_{N=1}\left[\left(1-\lambda e^{-2\pi i k/N}\right)^{3/2} - \left(1-\lambda e^{-2\pi i j/N}\right)^{3/2}\right], \tag{3.5}$$

while the corresponding wall tensions are

$$T_{jk} = |\mathcal{Z}_{jk}|\,. \tag{3.6}$$

Note that T_{kk+N} goes to zero at the branch points in the λ-plane [3], corresponding to the fact that the vacua labelled by $\mathcal{W}_k$ and $\mathcal{W}_{k+N}$ collide at these points.

3.2 Marginal stability

We now specialise to walls interpolating between two vacua on the same branch, and for definiteness restrict to walls associated with the central charges $\mathcal{Z}_{-kk}$. For sufficiently small λ, such configurations are naturally interpreted as bound states of $2k$ minimal 1-walls with charges $\mathcal{Z}_{pp+1}$ which interpolate between adjacent vacua. For this reason, we will generally focus on the simplest bound state $\mathcal{Z}_{-11}$ composed of two 1-walls. Supersymmetry demands that such putative BPS bound states, when present, are at least marginally bound. The corresponding submanifolds on which

$$T_{-11} = T_{-10} + T_{01} \tag{3.7}$$

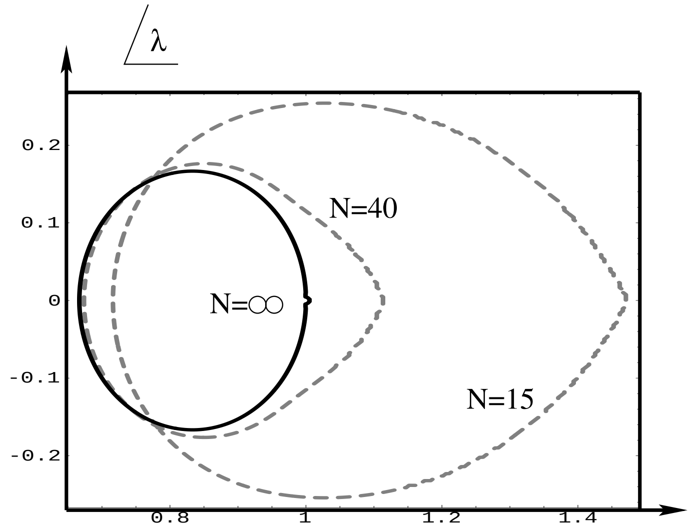

Figure 2: A plot of the CMS curves in the λ-plane for the 2-wall composite, and several values of N. For $N \to \infty$ the limiting form of the curve is a circle described by (3.14).

are of co-dimension one in the parameter space, and allow for possible discontinuities in the spectrum, where BPS bound states may delocalise and leave the spectrum. The position of these curves of marginal stability (CMS) in parameter space is fixed by supersymmetry and therefore is purely kinematic. It is this question that we will address in the remainder of this section, while the subsequent dynamical issue of whether discontinuities in the spectrum do actually occur on these submanifolds will be addressed subsequently.

A convenient characterisation of the submanifolds on which (3.7) holds is that the relative phase of the two constituent central charges vanishes,

$$\omega(\lambda)|_{\rm CMS} = 0\,, \tag{3.8}$$

where

$$e^{i\omega} \equiv \frac{\mathcal{Z}_{-10}\bar{\mathcal{Z}}_{01}}{|\mathcal{Z}_{-10}\mathcal{Z}_{01}|}\,. \tag{3.9}$$

Using the explicit form for the central charges (3.5), the condition (3.8) implies the following two (real) constraints:

$$\mathrm{Im}(f_1(\lambda)f_1(\bar{\lambda})) = 0\,, \quad \mathrm{sgn}(f_1(\lambda)f_1(\bar{\lambda})) < 0\,, \tag{3.10}$$

where

$$f_k(\lambda) \equiv 1 - \left(1 - 2i\frac{\lambda}{1-\lambda}e^{i\pi k/N}\sin\frac{\pi k}{N}\right)^{3/2}\,. \tag{3.11}$$

The CMS defined by the constraints in (3.10) is exhibited in figure 2 for several values of N; it defines a closed curve for $N > 6$, and has a simple limiting form for large-N that we will now describe in more detail.

As follows from (3.5), the expression (3.9) for the relative phase contains branch points at $\lambda = \{e^{-2\pi i/N}, 1, e^{2\pi i/N}\}$ internal to the λ-plane, and this branch structure is inherited by the equation (3.10) defining the CMS. Although having physical significance in signalling the presence of singular points connecting this branch with others associated with massless vacua, these points are not crucial here in that the spectrum of finite tension walls is at most discontinuous on curves of co-dimension one, and so the co-dimension two branch points can always be avoided.

To make use of this fact, we can simplify the structure of the CMS by taking N large, in a scaling limit such that

$$N\omega(\lambda) \;=\; \text{constant}\,, \tag{3.12}$$

whereupon at leading nontrivial order the branch points lie along a line, $\lambda_k \sim 1 \pm 2\pi i k/N$. In this regime it is straightforward to evaluate the relative phase $\omega(\lambda)$ for the $\mathcal{Z}_{-11}$ wall bound state. We obtain the following simple expression,

$$\omega(\lambda) = \frac{2\pi}{N}\left[1 - \frac{1}{2}\mathrm{Re}\left(\frac{\lambda}{1-\lambda}\right)\right] + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{3.13}$$

which reduces to $\omega = 2\pi/N$ in the limit $\lambda \to 0$ of pure SYM, and more generally is a harmonic function of λ, with a corresponding pole at the branch point $\lambda_0 = 1$ associated with the intermediate vacuum.

From this expression, we find that at large-N the phase ω vanishes on a circle in the complex λ-plane described by

$$\text{CMS}: \quad |6\lambda - 5|^2 = 1\,. \tag{3.14}$$

Note that since the curve only touches the line $\lambda = 1 + i\alpha$ at the point $\alpha = 0$, it is consistent to track the curve all the way to $\lambda = 1$ since it avoids the additional branch points at λ_{-1} and λ_1. There are apparently two distinguished points on the CMS, at $\lambda = 1$ and $\lambda = 2/3$, where the curve intersects the real axis. The former corresponds to the branch point for the intermediate vacuum, while the latter has an interesting interpretation within the Dijkgraaf-Vafa matrix model [6], that we will explore in more detail below.

3.3 The matrix model

The particularly simple large-N structure for the CMS determined above hints at a more transparent interpretation. One point of view that we will provide some evidence for below is that the onset of marginal stability in this regime is characterised by a qualitative change of the intermediate vacuum. For the composite state considered above, the relevant vacuum has a critical point at $\lambda = 1$, and presumably also intersects with massless vacuum branches at other points whose precise locations depend on N [3, 16].

The fact that the CMS passes through the critical point at $\lambda = 1$ is then not too surprising as the intermediate vacuum at this point will have additional massless excitations. The second intersection of the CMS with the real axis at $\lambda = 2/3$ is less immediately attributable to any pathology of the vacuum. However, it is a distinguished point within the geometric description of the vacua in the $\mathcal{N} = 2$ limit, and thus also within the matrix model picture.

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To illustrate this point, we first recall some of the relevant aspects of the Dijkgraaf-Vafa matrix model. In particular, for the theory at hand, there now exist some rather general arguments [6, 35, 16] implying that specific aspects of the chiral sector of the theory are captured by a holomorphic matrix model of the form

$$e^{-M^2\mathcal{F}(S)/S^2} = \int [dX] e^{-\frac{M}{S}\,\mathrm{tr}\,\mathcal{W}_{\mathrm{cl}}(X)}\,, \tag{3.15}$$

where X is an $M \times M$ matrix, and we take $M \to \infty$ to isolate the planar sector. S serves as a loop counting parameter in the planar limit of the matrix model, but ultimately, one makes the identification $S = -\frac{1}{32\pi^2}\,\mathrm{tr}\,W^\alpha W_\alpha$ within the gauge theory. The large-M saddle point is characterised by the condensation of the eigenvalues of X into a set of cuts $[a_k, b_k]$, each corresponding classically to one of the roots of $\mathcal{W}'_{\mathrm{cl}}(x) = 0$. The saddle point condition is conveniently expressed in terms of the force on a test eigenvalue,

$$y_{\mathrm{m}}(x) = \mathcal{W}'_{\mathrm{cl}} - 2S\mathcal{R}_m \qquad \text{where} \qquad \mathcal{R}_{\mathrm{m}} = \frac{1}{M}\left\langle \mathrm{Tr}\,\frac{1}{x - X}\right\rangle. \tag{3.16}$$

$\mathcal{R}_{\mathrm{m}}$ is the trace of the resolvent whose discontinuity across the cuts determines the eigenvalue distribution (see e.g. [36]). We are concerned here with 1-cut solutions, corresponding to classically unbroken U(N) vacua, and in terms of y_m the (large M) saddle point requires [6, 3],

$$y_{\mathrm{m}}^2 = (\mathcal{W}'_{\mathrm{cl}}(x))^2 - f_1(x) = g^2(x - x_*)^2(x - a)(x - b) \tag{3.17}$$

where f_1 is a polynomial of degree one, and a and b are the endpoints of the cut, with x_* a double point corresponding to the degenerate second cut. This constraint describes a degenerate hyper-elliptic curve, and is equivalent to the appropriate degeneration of the U(N) Seiberg-Witten curve [13, 14].

For this saddle point configuration, the gauge theory superpotential is determined by matrix model prepotential as [6]

$$\mathcal{W}_{\mathrm{m}} = N\Pi_B - (\tau + k)\Pi_A\,, \tag{3.18}$$

in the 1-cut sector, where the periods (Π_A, Π_B) are given in terms of the resolvent $\mathcal{R}_{\mathrm{m}}$ via

$$\Pi_A = 2\pi i S = \oint_A y_{\mathrm{m}}\,, \qquad \Pi_B = 2\pi i\frac{\partial\mathcal{F}}{\partial S} = \oint_B y_{\mathrm{m}}\,, \tag{3.19}$$

using a symplectic basis of (compact) A and (noncompact) B cycles for the surface (3.17), as shown in figure 3.

Making use of (3.17), the integrals in (3.19) may be expressed in terms of elementary functions. Evaluating Π_B in the limit $\Lambda_0 \to \infty$, and dropping an irrelevant constant, one finds that the contribution from Π_A in (3.18) serves to reconstruct the dependence on the dynamical scale $\Lambda_{N=2}$. With $p = p(S)$ denoting the mid-point of the cut, one finds that [6, 3, 16, 37],

$$\mathcal{W}_{\mathrm{m}}(S) = N\mathcal{W}_{\mathrm{cl}}(p(S)) + S\ln\left[\frac{e\,m(p(S))\Lambda_{N=2}^2}{S}\right]^N. \tag{3.20}$$

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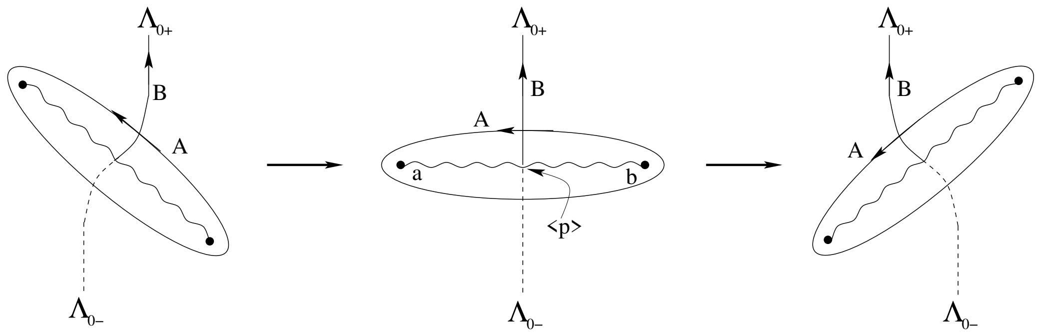

Figure 3: A conventional basis of A and B cycles for the 1-cut degeneration of the curve. The centre of the cut is identified with p, and the cut 'rotates' as shown in passing from one vacuum to another via a domain wall.

This result — a generalisation of the Veneziano-Yankielowicz [38] superpotential — is identical[4] to the answer one obtains from (2.4) by integrating in S via a Legendre transform with respect to $\ln \Lambda_{N=2}^{2N}$ [16], and then integrating out p in favour of S. On extremising $\mathcal{W}_{\rm m}(S)$, and with the implicit dependence on the scale $\Lambda_{N=2}$, the on-shell superpotential takes the form,

$$\mathcal{W}_{\rm m}|_k = N\Pi_B(\tau_k)\,, \tag{3.21}$$

where

$$\tau_k \equiv \frac{\partial \Pi_B}{\partial \Pi_A} = \frac{\tau + k}{N} \tag{3.22}$$

is the modular parameter of the noncompact elliptic curve, indicating that it is an N-fold cover of the bare curve. Of course, a renormalisation group-invariant version of this statement is that $\mathcal{W}_{\rm m}|_k$ depends on N only via the (complexified) dynamical scale

$$\Lambda_{N=2}^2 \exp\left(\frac{2\pi i k}{N}\right) = \Lambda_0^2 \exp\left(\frac{2\pi i(\tau + k)}{N}\right). \tag{3.23}$$

This explanation of the remarkably simple N-dependence of the vacuum values of the superpotential, and consequently of the central charges is one of the virtues of the matrix model approach, since the matrix model itself is N-independent, and results from the planar saturation of the chiral sector [6]. In particular, using the topological charges $n^k_{(ij)} = \delta^k_j - \delta^k_i$, the central charges can be compactly written in terms of the B-period as follows,

$$\mathcal{Z}_{ij} = 2n^k_{(ij)}\mathcal{W}_k = 2Nn^k_{(ij)} \oint_B y_m(\tau_k)\,. \tag{3.24}$$

Geometrically, the 1-cut configuration is unique, with the nontrivial vacuum structure arising via extremisation of (3.18) for a given value of k. However, for the purpose of interpreting domain walls in this picture, it is useful to note that the rotation of the cutoff $\Lambda_0 \to \Lambda_0 e^{2\pi i}$ induces the monodromy $\Pi_B \to \Pi_B - \Pi_A$ [37], and thus if we choose to fix the phase of the regulator Λ_0, we can view the process of passing through a wall as

[4]The on-shell equivalence between the 'equilibria' of the integrable Toda system and the matrix model saddle points has recently been verified more generally [32, 33].

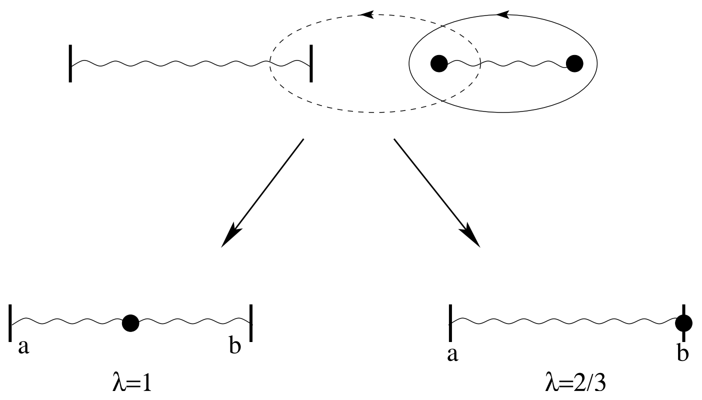

Figure 4: A schematic illustration of the degeneration of two specific cycles within the generic $\mathcal{N} = 2$ U(N) curve. We can interpret the point $\lambda = 1$ as corresponding to the degeneration of one of the cycles of the generic curve, while the point $\lambda = 2/3$ corresponds to the degeneration of two such cycles. In contrast, the 1-cut saddle-point for the matrix model is always characterised by one non-degenerate cycle and a double point at x_*. As shown, at $\lambda = 1$ the double point sits at the midpoint of the remaining cut, while for $\lambda = 2/3$ it sits at one end.

corresponding to a rotation of the cut as exhibited in figure 3, with the corresponding rearrangement of the eigenvalues.[5]

With this formalism in hand, we now reconsider the near-CMS regime. and in particular the intercept points $\lambda = 1$ and $\lambda = 2/3$. As follows from (3.17), both these points correspond to degenerations of the 2-cut solution as shown in figure 4. In the plot, for illustration, we show some of the additional cycles relevant within a 2-cut degeneration of the Seiberg-Witten curve. However, in the 1-cut solution we consider here, the only remnant of these degenerations is that at $\lambda = 1$, the double point at $x = x_*$ lies at the mid-point of the cut, while at $\lambda = 2/3$ the double point lies at one end. The degenerating cycles do not play a direct role in the matrix model solution.

The point $\lambda = 2/3$ is, however, known to correspond to a critical point within the $1/M$-expansion of the matrix model [3, 36], i.e. of the prepotential. This suggests that it should be possible to see consequences of the degeneration within (3.20). At $\lambda = 1$, this is of course reflected in the branch point for $\mathcal{W}_0(\lambda)$. Evidence of the degeneration at $\lambda = 2/3$ is less apparent, but can be uncovered if we define the following 'β-function' for the period of the curve (3.17),

$$\beta_k \equiv S \frac{\partial \tau(S)}{\partial S}\bigg|_k , \tag{3.25}$$

where the period (3.22) is given (off-shell) by $\tau(S) = \partial_S^2 \mathcal{F}$ in terms of the prepotential. Evaluating $\beta(\lambda)$ in the relevant $k = 0$ intermediate vacuum, we find

$$\beta_0 = 2N \frac{1-\lambda}{3\lambda - 2} , \tag{3.26}$$

[5]In the context of 'classical walls', where the cuts corresponding to the two vacua have distinct classical limits, it was noted by Dijkgraaf and Vafa that walls are apparently dual to eigenvalue tunnelling [6]. Such an 'instanton-like' interpretation seems more problematic here as the relevant vacua are classically degenerate.

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which vanishes for $\lambda = 1$, and diverges for $\lambda = 2/3$. This direct connection between degenerations of the geometry and the position of the CMS for large-N is rather suggestive, but without a clearer picture of the physical excitations associated with $\mathcal{W}_{\rm eff}(S)$ it is difficult to fully interpret the consequences of the latter singularity,[6] and we will not pursue this further here.

In the next section, we turn to the dynamical question of deducing the multiplicity of states, and indeed whether discontinuities do indeed arise on crossing the CMS curves discussed above.

4. BPS wall spectrum

The multiplicity of 1/2-BPS multiplets in $\mathcal{N} = 2$ theories in 1+1D with fixed boundary conditions is given by the CFIV index, ν_{jk}, formally defined as [8]

$$\nu_{jk} \equiv \mathrm{Tr}_{jk}\, F(-1)^F\,, \tag{4.1}$$

where F is the fermion number of the corresponding state. The crucial property of ν_{jk}, following from its definition, is its stability under various deformations of the theory. In particular, it was shown in [8] that it is stable under (nonsingular) variations of the D-terms, such as the Kähler potential, which are unconstrained by supersymmetry. This makes calculation of ν_{jk} tractable as we can deform the theory, for example via compactification as here, and the result can be shown to be independent of the compactification radius R which enters the Kähler potential.

Moreover, ν_{jk} is also stable under certain variations of the F-terms. For a Wess-Zumino model, characterised by a Kähler potential $\mathcal{K}(X^a,\bar{X}^a)$ and a superpotential $\mathcal{W}(X^a)$, as we have here this can be understood via a geometric reformulation of ν_{jk} [40]. Specifically, an important consequence of the Bogomol'nyi equation for 1/2-BPS walls is that the superpotential must trace out a straight line in the $\mathcal{W}$-plane [41, 42]; i.e.

$$\mathcal{W}(X^a) = (1-t)\mathcal{W}_0 + t\mathcal{W}_1\,, \tag{4.2}$$

where $t \in [0,1]$ is a fiducial variable parametrising the transverse coordinate to the wall, and $\mathcal{W}_0$ and $\mathcal{W}_1$ are the corresponding vacua between which the wall interpolates. The set of solutions to the linearised Bogomol'nyi equation about each vacuum defines a cycle $\triangle_j$ in field space, and one can show [40] that by comparing the two cycles $\triangle_j$ and $\triangle_k$ at a given value of $t \in [0,1]$, the number of solutions is given precisely by the intersection number,

$$\nu_{jk} = \triangle_j \circ \triangle_k\,. \tag{4.3}$$

The topological character of the intersection number then implies that ν_{jk} is stable under deformations of $\mathcal{W}$ which do not lead to additional vacua crossing the straight line trajectory between $\mathcal{W}_0$ and $\mathcal{W}_1$, as it is only in this case that the intersection number varies according to a Picard-Lefschetz monodromy [40].

[6] As noted recently [39], another quantity with the same characteristics is the formal 'mass term' $\mathcal{W}''_{\rm eff}(S)$ which, however, is again difficult to interpret without knowledge of the Kähler potential.

Thus, to prove the existence of 1/2-BPS states it is sufficient to verify the presence of smooth profiles in field space for which the interpolating trajectory in the punctured $\mathcal{W}$-plane — with additional vacua excised — is homotopic to the straight line (4.2). With this argument in mind, and given our knowledge of the structure of the CMS curves, our strategy will be first to determine the spectrum at a convenient point — we will consider the pure SYM regime $\lambda \to 0$ — and then to study whether this multiplicity is preserved on crossing the CMS. For the latter test, we construct the leading order interaction potential, as a function of λ, between the constituents of the 2-wall composite discussed in the previous section.

4.1 Multiplicity for $\lambda \ll 1$

We will first consider the spectrum at a special point $\lambda = 0$, where the infrared theory reduces to pure $\mathcal{N} = 1$ SYM. BPS walls in this theory have been studied from several points of view [43]–[48]. Within the low energy description on $\mathbb{R}^3 \times S^1$, the effective superpotential reduces in the $\lambda \to 0$ limit to the following (complexified) affine-Toda form [21, 22, 23, 24, 25, 11, 26, 27, 28],

$$\mathcal{W}_{\text{eff}}(\lambda = 0) = \Lambda^3_{\mathcal{N}=1} \sum_{a=1}^{N} x_a \,, \tag{4.4}$$

subject to the constraint that $\prod x_a = 1$. Using the cyclic symmetry, we choose the initial vacuum to be $x_a = 1$ and the second vacuum to be $x_a = e^{2\pi i k/N}$, with $k < N$. Within the wall, the winding number of each field must then vary from zero to k/N mod $\mathbb{Z}$, and moreover the constraint $\prod x_a = 1$ implies that at any particular point along the trajectory (4.2) the winding numbers of the fields x_a must sum to zero. Thus, at any non-vacuum point, they cannot all be equal. To minimise gradient energy, there are only two allowed winding numbers: k/N and $k/N - 1$ [49], and this leads us to the following resolution of the constraint in terms of a new variable y [50],

$$x_{a=1,\ldots,k} = y^{N-k} \,, \qquad x_{a=k+1,\ldots,N} = y^{-k} \,. \tag{4.5}$$

The relation (4.2) then implies that for the corresponding k-walls

$$k y^{N-k} + (N-k) y^{-k} = N(1-t) + N t e^{2\pi i k/N} \,. \tag{4.6}$$

For $N = 2$ (and thus $k = 1$), this relation can be straightforwardly inverted to obtain the interpolating trajectories. However, for generic N, we can instead simply verify the existence of BPS solutions by following an approach used in the context of the $\mathcal{N} = 2$ affine-Toda model in 1+1D[7] [50]. Note that since y is a complex variable, there are two linearised solutions about each vacuum, but at most one combination of these can link

[7] As an aside, bosonic affine-Toda theory in 1+1D with an imaginary coupling is also known to exhibit solitons [51]. In general, these solutions satisfy a second-order equation with the potential (4.4), and so are not related to the configurations we are concerned with here. However, the case $N = 2$ is an exception where the sine-Gordon soliton is (up to certain rescalings) a BPS solution if we use the classical Kähler metric. One may understand this via noting that affine-Toda solitons also satisfy a set of first-order (Bäcklund) equations [52] which reduce precisely to the Bogomol'nyi equations for $N = 2$.

to form a BPS trajectory. To verify that such a trajectory exists we follow the argument described above [40, 50]. In particular, with the ansatz $y(t) = e^{2\pi i t/N}$, we find

$$|\mathcal{W}_{\text{eff}}(\lambda = 0)| = |N - k + k e^{2\pi i t}| \leq N\,, \tag{4.7}$$

where the final equality holds only at the initial and final vacua. Therefore, since the remaining vacua lie on a circle of radius N, this suffices to prove the existence of the corresponding BPS multiplet [50]. Moreover, accounting for the permutation symmetry in the identification of y, the full multiplicity is given by

$$\nu_{0k}(\lambda = 0) = \begin{pmatrix} N \\ k \end{pmatrix}. \tag{4.8}$$

This result, conveniently interpreted as a multiplet transforming in the k^{th} fundamental representation of SU(N), is consistent with other calculations of the degeneracy of BPS walls in pure $\mathcal{N} = 1$ SYM [53, 49].

4.2 Inter-wall potential for generic λ

To explore how the multiplicity of BPS states varies as we move in parameter space, it is useful to follow a somewhat different approach. It should be clear from the analysis above that the spectrum of 1-walls will be stable for $|\lambda| < 1$ where the structure of neighbouring vacua does not change qualitatively. However, it is less apparent that the multiplicity of higher k-walls is stable in this region of the parameter space. We can determine possible discontinuities by treating k-walls as bound states of k 1-walls and calculating the inter-wall potential as a function of λ. In practice, it will be sufficient to focus on the simplest case of a 2-wall bound state, corresponding to the example considered in the previous section.

To proceed, we note that on general grounds the only global symmetries broken by the wall configurations are super-translations. We then infer that the only bosonic moduli of a system of two asymptotically separated 1-walls will be their respective centre-of-mass positions. It is an implicit assumption that any other worldvolume fields are massive and can be ignored for the purpose of considering the ground state. The bosonic moduli space of the constituent system is then $\mathbb{R}^2$, and we can always decouple the overall translational mode, and thus the relative moduli space is one-dimensional, parametrised by a real field r. The problem at hand then reduces to computing the potential induced on this space when the constituent 1-walls are at a finite separation.

For the region $|\lambda| < 1$, it convenient to further restrict the problem by considering the regime where $R\Lambda_{N=2} \ll 1$ which means that the adjoint scalar fields p_a are (generically) of higher mass than the gauge modes q_a and can be integrated out, leading to the reduced superpotential,

$$\mathcal{W}_{\text{eff}}(x) = \frac{2N}{3\lambda}\Lambda^3_{N=1}\left(1 - \frac{1}{N}\sum_{a=1}\left(1 - \frac{\lambda}{2}\left(x_a + x_{a-1}\right)\right)^{3/2}\right), \tag{4.9}$$

which is now a function only of x_a. However, the presence of branch points implies that this reduced system is valid only for sufficiently small λ, for which the field profiles remain well away from the cuts.

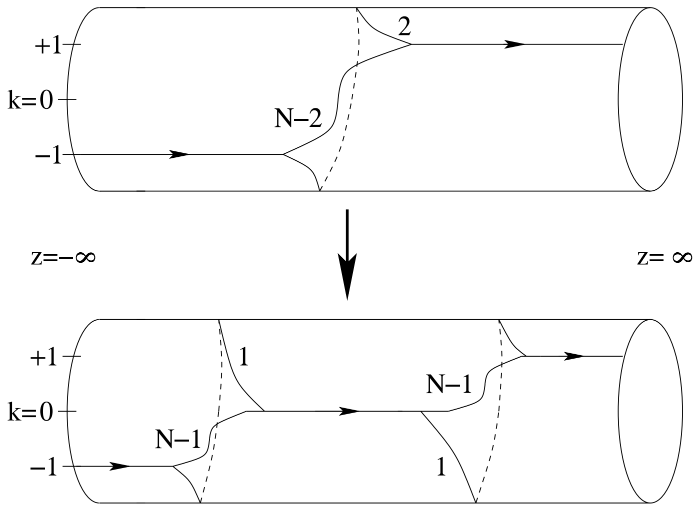

Figure 5: A schematic representation of the profile of the phase of the x_a fields within the 2-wall bound state in the upper plot. $N-2$ of the components have winding number $2/N$, while the remaining two have winding number $2/N-1$. In the lower plot the two 1-wall constituents are separated so that the fields all lie near the intermediate vacuum in the centre of the 2-wall. This configuration is unstable but can be used to extract the leading order interaction potential.

As noted above, we focus on 2-walls and fix the two vacua to be $x_a^{(0)} = e^{-2\pi i/N}$ and $x_a^{(1)} = e^{2\pi i/N}$ as in section 3. The putative 2-walls in this sector, counted by ν_{-11} can be viewed as bound states formed from the 1-walls for which $\nu_{-10} = N$ and $\nu_{01} = N$. A schematic illustration of how the phases of the x_a fields must vary over the wall profile is shown in figure 5.

The leading order potential between the constituents can be determined by expanding the tension of wall two in the background of wall one (see also [54, 55]). More precisely, we can write the tension of the second wall $T_2 = 2|\mathcal{W}_{(1)} - \mathcal{W}_{(0)}|$ as

$$T_2(r) = T_2 - 2\mathrm{Re}\left[e^{-i\gamma}\delta^a_{(-10)}(z_0)\delta^b_{(01)}(z_0+r)\partial_a\partial_b\mathcal{W}_{(0)}\right] + \cdots , \tag{4.10}$$

where the second term is the leading order interaction potential $V(r)$ between the constituent 1-walls positioned at $z = z_0$ and $z = z_0 + r$ respectively (see figure 5), with z the transverse coordinate. In this expression, the subscripts label the phase of x_a in the relevant vacua, and $\delta^a = x^a - x^a|_{(0)}$ are the deviations of each field from its value in the intermediate vacuum, while γ is the phase of $\mathcal{Z}_{01}$. These fields satisfy the linearised Bogomol'nyi equations given by

$$\partial_z \delta^a = e^{i\gamma} M^a_b \bar{\delta}^b , \qquad \text{where} \qquad \sum_{a=1}^{N} \delta^a = 0 , \tag{4.11}$$

and $M^a_b = g^{ac}\partial_c\partial_b\mathcal{W}_{(0)}$ is the (complex) mass matrix in the intermediate vacuum, with g_{ab} the Kähler metric. The additional constraint on the perturbations follows from $\prod x_a = 1$.

The Bogomol'nyi equations decouple in the mass-eigenstate basis for the intermediate vacuum, where M is diagonal. In other words, since M is symmetric, the eigenvalues of the following system [40]

$$\begin{pmatrix} 0 & e^{-i\gamma}\bar{M} \\ e^{i\gamma}M & 0 \end{pmatrix} \begin{pmatrix} \delta \\ \bar{\delta} \end{pmatrix} = \pm m_\mu \begin{pmatrix} \delta \\ \bar{\delta} \end{pmatrix}, \tag{4.12}$$

are real and paired $\{m_\mu, -m_\mu\}$, for $\mu = 1, \ldots, N-1$, and moreover $\{m_\mu\}$ can be identified with the mass eigenvalues in the intermediate vacuum. We deduce that the complex phase of the perturbations δ arises purely from the integration constant, and can be fixed via comparison with (4.11).

With explicit solutions for the linear perturbations in hand, it is convenient to first use the Bogomol'nyi equation to write the potential in the form

$$V(r) = -2\mathrm{Re}\left[\delta_{(-10)}(z_0) \cdot \bar{v}_{(01)}(z_0 + r)\right] + \cdots, \qquad \text{where} \quad v^a = \partial_z \delta^a, \tag{4.13}$$

and the inner product is that associated with the Kähler metric. On substituting the appropriate solutions, we can read off the leading order potential,

$$V(r,\lambda) = -2\sin\left(\frac{\omega(\lambda)}{2}\right) \sum_{\mu=1}^{N-1} m_\mu e^{-m_\mu r} + \cdots, \tag{4.14}$$

where $\omega(\lambda)$ is the relative phase between the central charges of the two constituent 1-walls, as defined in (3.9).

Note that the constraint apparent in the fundamental weight basis in (4.11) is reflected in the mass spectrum. For example, in the limit $\lambda = 0$, we find

$$M^a_b(\lambda = 0) \propto \hat{C}^a_b \quad \Longrightarrow \quad m_\mu(\lambda = 0) \propto \sin^2 \frac{\pi\mu}{N}, \tag{4.15}$$

where $\hat{C}^a_b$ is the Cartan matrix for affine SU(N). The low-lying mass eigenvalues then appear in pairs, and depend nontrivially on N.

It is now clear that the presence of the corresponding 2-walls depends purely on the value of $\omega(\lambda)$, which is kinematic in nature as one should expect, since discontinuities in the BPS spectrum are allowed only when the corresponding bound states are marginally stable, namely when $\omega(\lambda) = 0$. Near these submanifolds the potential reduces to

$$V(r,\lambda) = -\omega(\lambda) \sum_\mu m_\mu e^{-m_\mu r} + \cdots, \tag{4.16}$$

which is linear in ω and thus a discontinuity in the spectrum should apparently occur on crossing the curve. Moreover, the potential applies to all components of the multiplet (4.8), and so it would seem that the entire multiplet is removed from the spectrum when $\omega(\lambda) < 0$. However, before reaching this conclusion, we first need to consider the effect of quantum corrections.

4.3 Quantum corrections

Before drawing conclusions regarding discontinuities in the spectrum, it is important to understand the status of the leading order potential that we have derived. In particular, we are interested in submanifolds in parameter space on which the bound state is marginally stable, where $\omega(\lambda) = 0$, and the arguments above suggest that we can equivalently define the submanifold via the relation $V(r, \lambda) = 0$. However, it turns out that the latter relation is not stable to quantum corrections, and more work is required. In actual fact quantum corrections, associated with the fermionic content of the model, are very important and become increasingly so near the CMS [54]. Fortunately, one finds that the qualitative insight drawn from $V(r, \lambda)$ is nonetheless correct [54]. Indeed, this must be the case on the grounds that the condition $\omega(\lambda) = 0$ describing the CMS is kinematic and protected by supersymmetry.

To proceed, we will compactify the spatial worldvolume coordinates on an additional circle of sufficiently small radius L. The low energy description of the worldvolume theory, i.e. for scales well below $1/L$, then reduces to quantum mechanics on a one-dimensional moduli space coordinatised by r. In this regime, the leading order potential (4.16) is corrected by the fact that walls are 1/2-BPS, and thus the (dimensionally reduced) worldvolume theory must possess $\mathcal{N} = 2$ worldline supersymmetry. Such quantum mechanical systems are well understood [56]. If we denote by $\{Q_1, Q_2\}$ the supercharges which act trivially on the putative BPS bound state, i.e.

$$(Q_1)^2 = (Q_2)^2 = (RL)(T - |Z|) \ll (RL)T \,, \tag{4.17}$$

we expect to find a bound state if this system exhibits a unique vacuum; the multiplet structure is then reproduced on tensoring this state with the free centre-of-mass sector.

The generic structure of $\mathcal{N} = 2$ SQM was first described by Witten [56], and we can realise the algebra as follows in terms of a quantum mechanical superpotential $\mathcal{W}_{\rm QM}(r)$,

$$\begin{aligned} Q_1 &= \frac{1}{\sqrt{2M_r}} \left[\pi_r \sigma_1 + \mathcal{W}'_{\rm QM}(r)\sigma_2\right] , \\ Q_2 &= \frac{1}{\sqrt{2M_r}} \left[\pi_r \sigma_2 - \mathcal{W}'_{\rm QM}(r)\sigma_1\right] , \end{aligned} \tag{4.18}$$

where $M_r = (RL)(1/T_1 + 1/T_2)^{-1}$ is the effective reduced mass. The hamiltonian is then given by

$$\mathcal{H}_{\rm QM} = M - |Z| = \frac{1}{2M_r} \left[\pi_r^2 + (\mathcal{W}'_{\rm QM}(r))^2 + \sigma_3 \mathcal{W}''_{\rm QM}(r)\right] , \tag{4.19}$$

where the second term is the classical potential, and the final term is a quantum correction of $\mathcal{O}(\hbar)$.

We have not explicitly allowed any nontrivial corrections to the kinetic term. In general one would expect that massive exchanges will introduce such terms, i.e. the coefficient of π_r^2 should take the schematic form $g_{rr} = 1 + \mathcal{O}(e^{-mr})$. However, by a rescaling, g_{rr} can always be absorbed into the potential, and this is implicitly the basis used here since we will in practice match to the full bosonic potential $V(r) = g^{rr}(\mathcal{W}'_{\rm QM})^2$.

Identifying the quantum mechanical superpotential $\mathcal{W}_{\rm QM}(r)$ in general requires a detailed analysis of the model at hand, as in [54]. However, we can easily extract the general structure near the CMS by comparison with the leading order classical potential in (4.16) [57]. Having compactified the theory on T^2, we can augment (4.16) with the leading constant term corresponding to the binding energy of the two constituents. Near the CMS, the bosonic potential then has the generic form,

$$V(r,\lambda) = \frac{1}{2}M_r\omega(\lambda)^2 - \omega(\lambda)\sum_{\mu} m_\mu e^{-m_\mu r} + \cdots . \tag{4.20}$$

Comparing (4.19) with (4.20), we find

$$\mathcal{W}_{\mathrm{QM}}(r,\lambda) = \omega(\lambda)(M_r r) + \sum_{\mu} \exp(-m_\mu r) + \cdots , \tag{4.21}$$

and thus near the CMS,

$$V_{\mathrm{QM}}(r,\lambda) = \frac{1}{2}M_r\omega(\lambda)^2 - \sum_{\mu} m_\mu \left[\omega(\lambda) + \sigma_3\frac{m_\mu}{2M_r}\right]\exp(-m_\mu r) + \cdots . \tag{4.22}$$

The additional quantum correction to the leading term becomes important near the CMS where it remains finite.

Despite this correction, the classical equilibrium position survives as the maximum of the ground state wavefunction, which is easily determined from (4.18),

$$|\Psi_0\rangle = \exp(-\mathcal{W}_{\mathrm{QM}}(r))|-\rangle \stackrel{r\to\infty}{\longrightarrow} \exp(-\omega(\lambda)M_r r)|-\rangle + \cdots \tag{4.23}$$

where $|\pm\rangle$ are the eigenvectors of σ_3. In the limit $r \to \infty$ we have extracted only the leading exponential behaviour. It is apparent that this wavefunction is normalisable on only one side of the CMS, namely for $\omega > 0$, which determines the existence domain for the BPS bound state. The intuition regarding discontinuities in the spectrum drawn from (4.16) therefore survives at the quantum level, although the details are somewhat different [54, 57].

4.4 Discontinuities in the BPS spectrum

The analysis of the leading order potential allows us to reduce questions about the 2-wall multiplicity to purely kinematic issues concerning the dependence of the central charges on λ. In particular, as discussed in section 3, discontinuities only occur on co-dimension one submanifolds of the parameter space — curves of marginal stability — where $\omega(\lambda) = 0$. The analysis of this section has verified that such discontinuities do indeed occur in this case.

Following our discussion in section 3, it is convenient for illustrative purposes to take N large, so as to resolve the branch structure of the central charges as functions over the λ-plane. To leading order in $1/N$, we recall that the angle ω takes the simple form

$$\omega(\lambda) = \frac{2\pi}{N}\left[1 - \frac{1}{2}\mathrm{Re}\left(\frac{\lambda}{1-\lambda}\right)\right] + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{4.24}$$

and one readily verifies that ω is positive, and thus the BPS 2-wall bound states indeed exist outside the closed circle defined by $|6\lambda - 5|^2 = 1$, while the potential becomes repulsive, and the BPS bound states disappear from the spectrum in the interior shaded region of figure 6.

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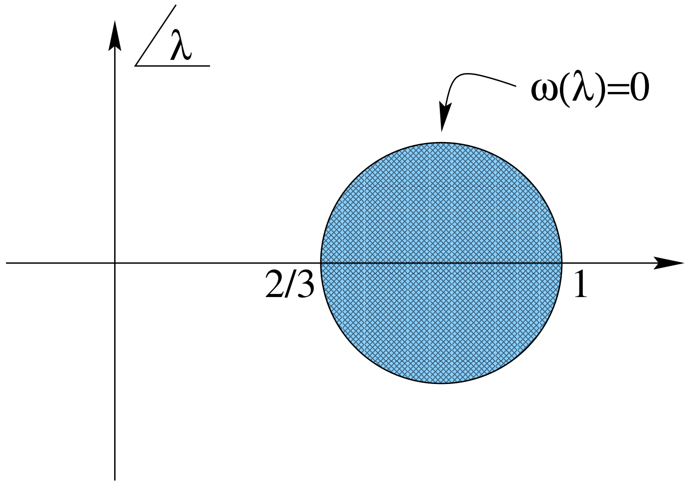

Figure 6: A plot of the circular (large-N) CMS curve in the λ-plane for the 2-wall composite. The 2-walls are absent in the shaded interior region.

In concluding this section, we will attempt to draw one further conclusion from these arguments relevant to the $\lambda \sim 1$ regime, although this is strictly outside the regime of validity of the dynamical approach being used. Indeed, it is apparent that the expression for the relative phase at large-N is singular at $\lambda = 1$. This is the critical point at which the intermediate vacuum $\mathcal{W}_0$ collides with the corresponding confining vacuum $\mathcal{W}_N$ on the second branch, signalling the presence of additional light states. Moreover, for N-even, $\lambda = 1$ is also a Seiberg-Witten singularity and denotes a point where the confining branch intersects the Coulomb branch.

For the composite 2-walls considered above, this suggests the possibility of forming a sandwich-like configuration where the outer regions are in confining phases — each associated in the near $\mathcal{N} = 2$ regime with condensation of charge-one dyons and anti-dyons respectively — while the interior domain, between the two constituent 1-walls, is in a massless phase at $\lambda = 1$. Such configurations would be of interest as there are then generic arguments for the formation of flux-tubes [34], i.e. open strings ending on the wall, and other features reminiscent of D-branes (see e.g. [58]). However, although not strictly valid for $\lambda = 1$, the arguments above suggest that, at least within this system, there is no such BPS configuration. i.e. as we tune $\lambda \to 1$ along a trajectory outside, but close to, the CMS the composite delocalises and on reaching $\lambda = 1$ only the massless phase remains.

5. Discussion

We have presented a limited exploration of the spectrum of BPS domain walls within $\mathcal{N} = 1$ gauge theories with adjoint matter, and specifically those characterised by a space of parameters associated with a classical polynomial superpotential. We have been far from exhaustive and focused on a small subset of states which in an appropriate limit reduce to the BPS walls present in $\mathcal{N} = 1$ SYM. The aim was simply to exhibit some of the features of the spectrum which, through the interpretation of the wall multiplicity in terms of the CFIV index, provides 'chiral data' on the parameter space.

In this final section, we will return to some of the original motivations noted in section 1, and comment on some extensions.

5.1 Embedding within $\mathcal{N} = 1^*$

One motivation for studying the BPS spectrum was to gain insight into additional symmetries (or dualities) acting on the parameter space. With the aim of gaining a clearer picture of the quantum symmetries preserved by the BPS spectrum, it is helpful to consider the natural UV completion of the theory studied here within $\mathcal{N} = 4$ SYM, a theory which exhibits S-duality. Perturbing $\mathcal{N} = 4$ SYM via the addition of a cubic superpotential for the three adjoint chiral fields formally preserves modular invariance [59, 60] if we assign the mass terms modular weight $(-5/6, 1/6)$, and the cubic couplings weight $(-1, 0)$. One may then construct the low energy effective superpotential by compactifying on $\mathbb{R}^3 \times S^1$ [11], and using the correspondence with the elliptic Calogero-Moser integrable system [61, 10]. In particular, the gauge invariant monomials are again identified with the action variables, $\langle \mathrm{tr}(\Phi^p)\rangle \longleftrightarrow \mathrm{tr}(L^p)$, where L is the corresponding Lax matrix [62].

Suppressing the details, we can follow a similar procedure to that presented in section 2 to determine the massive vacua. Restricting to those which correspond to a classically unbroken $\mathrm{U}(N)$ gauge group, the central charges have the form [63],

$$\mathcal{Z}_{ij} = 2\sum_k n_k^{(ij)} \mathcal{W}_k = \frac{4N}{3\lambda}\Lambda^3_{N=1}\left[(1-\xi X_j)^{3/2} - (1-\xi X_i)^{3/2}\right], \tag{5.1}$$

where

$$\xi = 8g^2\frac{M^2}{m^2}, \tag{5.2}$$

in terms of the mass parameters for the three chiral fields $m_i = (m, M, M)$, with $m \ll M$, and

$$X_k = \frac{1}{24}\left[C(\tau) - \frac{p}{q}E_2\left(\frac{p\tau + k}{q}\right)\right], \tag{5.3}$$

where $(p, q) = (1, N)$, with $k = 0, \ldots, 2N-1$ for the confining vacua, and $(p, q) = (N, 1)$, with $k = 0, N$ for the Higgs vacuum. E_2 is the 2nd Eisenstein series, and $C(\tau)$ is a vacuum-independent constant. With an appropriate choice of $C(\tau)$, which corresponds to a particular basis for the action variables of the Calogero-Moser system, the combination ξX_k is modular invariant up to permutations of the vacua, and so the central charges have weight $(-1/2, 1/2)$. Consequently, the wall tensions are invariant up to permutation, and $\mathrm{SL}(2, \mathbb{Z})$ thus acts via permutations on the wall spectrum.

The parameter space in this theory is now two-dimensional, coordinatised (for the massive $\mathrm{U}(N)$ vacua) by $\{\xi, \tau\}$. One can show that in the decoupling limit, $M \to \infty$, the spectrum of 2-walls interpolating between confining vacua exhibits the same CMS curve as found above, but whose profile is (not surprisingly) corrected by fractional instanton effects. The permutation symmetry of the vacua under rotations of λ is now seen to follow directly from the symmetry under T-translations, namely $\tau \to \tau + 1$, in $\mathrm{SL}(2, \mathbb{Z})$. A more complete analysis of the spectrum and symmetries, including the 'classical' walls connecting confining and Higgs vacua, will appear separately [63].

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5.2 On the anomaly multiplet

Another motivation for this work was to try and understand whether the matrix model approach could be used to extract the chiral data associated with the BPS spectrum — namely to compute the CFIV index – in a purely four-dimensional context. An important point to recognise is that, while the confining vacua are described by 1-cut solutions to the matrix model, this is insufficient to describe BPS states. Indeed, the field profiles required to construct domain walls will necessarily pass through generic regions in moduli space where, in the $\mathcal{N}=2$ limit, the gauge group is maximally broken. This implies that we require knowledge of the large M solution to the matrix model with the maximal number of cuts. On reflection, this should not be too surprising on comparison with the approach used here, where we made use of the full $\mathcal{N}=2$ Seiberg-Witten curve.

One may then enquire as to how the nontrivial wall multiplicity, described in section 4 [53, 49], arises in the pure $\mathcal{N}=1$ SYM limit where, having integrated out the adjoint field, the effective superpotential $\mathcal{W}=\mathcal{W}(S)$ depends only on a single gluino condensate field S. Even discounting non-analyticities, a superpotential depending on a single field can at most describe unique interpolating solutions, and this is insufficient to explain the multiplicities found in section 4. One may then ask how the information about this multiplicity is encoded in four dimensions, if at all? To this end, it is worth recalling the resolution of a (perhaps) analogous puzzle that arises on compactification to 1+1D on a 2-torus (see also [17]). Since we are computing the index $\mathrm{tr}\, F(-1)^F$, one might anticipate that such an additional reduction should not affect the conclusions.

We can ignore the presence of the adjoint chiral field, so the reduced superpotential follows directly from a reduction of the 3D affine-Toda superpotential,

$$\mathcal{W}_{\rm 2D}(\Sigma, v_a) = \Lambda_{2D}\sum_{a=1}^{N} e^{-v_a} + \Sigma\left(\sum_a v_a\right), \tag{5.4}$$

where $x_a = e^{-v_a}$, with $\Lambda_{2D} = 4\pi^2(RL)\Lambda^3_{N=1}$, and we have inserted a Lagrange multiplier to explicitly enforce the decoupling of the overall U(1) factor. One can recognise this as the 1+1D mirror [65] of the $\mathcal{N}=2$ $\mathbb{C}P^{N-1}$ sigma model where we interpret Σ as the U(1) field strength within the corresponding linear sigma model. This relationship is also consistent with a direct reduction of pure SYM, since the moduli space of flat SU(N) connections on a torus is also $\mathbb{C}P^{N-1}$ [64, 65].

If we now integrate out Σ from (5.4), we simply enforce the decoupling of the central U(1), and we can reproduce the vacuum structure and nontrivial kink multiplicity in the same manner as described in section 4 [50]. However, we could also choose to integrate out the v_a's, or equivalently, the homogeneous $\mathbb{C}P^{N-1}$ chiral fields. From (5.4), we can do this at tree level to find,

$$\mathcal{W}_{\rm 2D}(\Sigma) = \Sigma\left(\ln\frac{\Lambda^N_{2D}}{\Sigma^N} + N\right). \tag{5.5}$$

This reduced theory describes, as it must, the same vacuum structure but, due to Gauss law, actually cannot describe any nontrivial kinks as one must satisfy the constraint $\Sigma|_{-\infty} = \Sigma|_{+\infty}$ [66]. As noted by Witten, the resolution of this puzzle is that the kinks are actually charged under the additional $\mathbb{C}P^{N-1}$ fields that have been integrated out, and one must account for the coupling of Σ to the corresponding current [66, 67]. In this sense, the superpotential (5.5), while sufficient for describing the vacua, is insufficient to fully describe the BPS spectrum.

Although there is no direct analogue of the Gauss law constraint in 3+1D, where we lift (5.5) to the Veneziano-Yankielowicz superpotential [38], one could make quite a close analogy if we were to interpret S as the field strength of a linear multiplet [68] containing a 3-form. While we will not pursue this further here, this issue is intriguing as these kinks are precisely the dimensional reduction of the BPS wall configurations that we have been counting.

Acknowledgments

I would like to thank Jan de Boer, Mikhail Shifman, and Arkady Vainshtein for helpful discussions and/or comments on the manuscript, and especially Nick Dorey for discussions and correspondence, some years ago, on BPS walls in compactified gauge theories.

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